Counting substructures of highly symmetric structures

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Overview

Counting orbits

- In the 1970s, Peter Cameron starting counting orbits for a group acting on a countable set.
- An action of G on a set X induces an action elementwise on the set of n-subsets for each $n \in \mathbb{N}$.
- The *growth rate* of the action, denoted $f_G(n)$, counts the orbits on n-subsets for each n.
- For the rest of the talk, we will assume $f_G(n)$ is always finite.
- See [3] for a survey.

COUNTING SUBSTRUCTURES

- Given G acting on X as in the previous slide, we may find a relational structure M such that the action of G on X is (essentially) the same as the action of $Aut(M)$ on M.
- Furthermore, M is *homogeneous*, i.e. every isomorphism between finite substructures extends to an automorphism of M.
- Then $f_G(n) = f_{Aut(M)}(n)$ also counts the number of (unlabelled) substructures of M of size n, up to isomorphism.
- We will use $f_M(n)$ in place of $f_{Aut(M)}(n)$.
- **Main thesis:** Slow growth rate should correspond to structural simplicity of M.

EXAMPLES

- Given a structure M, we let $\mathcal{C}_M$ be the class of finite substructures of M.
- $M = (\mathbb{Q}, \leq)$. Then $\mathcal{C}_M$ is the class of finite linear orders, and $f_M(n) \equiv 1$.
- M is a structure whose domain is partitioned into two unary relations. Then $\mathcal{C}_M$ consists of finite structures whose domain is split into red and blue points, and $f_M(n) = n$.
- M is an equivalence relation with infinitely many classes, each infinite. Then $\mathcal{C}_M$ is the class of finite partitions, and $f_M(n) \approx e^{\sqrt{n}}$.
- $\mathcal{C}_M$ consists of the leaves of full binary trees, and $f_M(n) = Catalan(n-1) \approx 4^n$.
- $\mathcal{C}_M$ consists of all finite graphs, and $f_M(n) \approx 2^{n^2/2}$.

The main theorem

- We give a detailed description of spectrum of growth rates slower than every exponential (in fact slower than ϕ^n), confirming some conjectures of Peter Cameron and Dugald Macpherson.

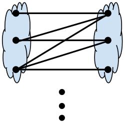
Theorem ([2])

Let M be a countable homogeneous structure. If $f_M(n) = o\left(\frac{\phi^n}{poly(n)}\right)$ for every polynomial (with $\phi \approx 1.618$), then $f_M(n) = o(c^n)$ for every $c > 1$. Furthermore, one of the following holds.

1. *There are $c > 0$, $k \in \mathbb{N}$ such that $f_M(n) \sim cn^k$.*
2. *There are $c > 0$, $k \in \mathbb{N}$ such that $f_M(n) = e^{\left(\Theta\left(n^{1-\frac{1}{k}}\right)\right)}$.*
3. *Let $\log^r(n)$ denote the r-fold iterated logarithm. There are $c > 0$ and $k, r \in \mathbb{N}$ such that $f_M(n) = e^{\left(\Theta\left(\frac{n}{(\log^r(n))^{1/k}}\right)\right)}$.*

STABILITY

- Originally defined for asymptotic enumeration of classes of infinite structures.
- A structure M is *stable* if it does not encode an infinite linear order, or equivalently, does not encode an infinite half-graph.



- What does "encode" mean?
- (Roughly) an induced subgraph of a graph definable from M.

Reducing to the stable case

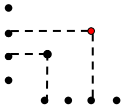
Theorem (Simon, 2018)

If M has growth rate slower than ϕ^n, then there is a stable structure M^* with the same growth rate.

- M^* is obtained by "forgetting" the orders on M.
- For example, if $M = (\mathbb{Q}, \leq)$, then $M^* = (\mathbb{Q}, =)$, and $f_M(n) = f_{M^*}(n) \equiv 1$.
- Stable structures are well-understood, and in particular have a well-behaved notion of independence.
- For example, linear independence in vector spaces, or "being in different connected components" in graphs of bounded degree.

MONADIC STABILITY

- M is *monadically stable* if it remains stable after an arbitrary coloring of its elements, with any number of colors.
- This can be characterized by behavior of the independence relation.
- If M is stable but not monadically stable, it encodes arbitrary bipartite graphs after a coloring, so has superexponential growth rate.
- Proof: Use the independence relation to find a grid.



MONADIC STABILITY CONTD.

- If M is monadically stable, then the independence relation is very well behaved.
- So M can be decomposed into a tree of substructures, which are all relatively independent from each other.
- This is similar to tree-decompositions in structural graph theory.
- Simple example: Decomposing a bounded-degree graph into connected components.
- Using this, Lachlan classified the homogeneous monadically stable structures.
- Their growth rates can be estimated fairly directly.

Structures with polynomial growth

- A structure has *depth 1* if it consists of infinitely many copies of a finite structure.

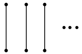


Figure: A depth 1 graph

- We also allow uniform interaction between copies, and a finite exceptional set.

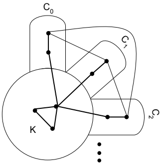


Figure: Another depth 1 graph

Structures with subexponential growth

- A structure has *depth 2* if consists of infinitely many copies of a depth 1 structure, with limited, uniform interaction between copies.
- Example: An equivalence relation with infinitely many classes, each infinite.
- This can be iterated to define depth d.
- Depth 2 corresponds to growth $f_M(n) = e^{\left(\Theta\left(n^{1-\frac{1}{k}}\right)\right)}$.
- Depth $d \geq 3$ corresponds to growth
  $$f_M(n) = e^{\left(\Theta\left(\frac{n}{\left(\log^{d-2}(n)\right)^{1/k}}\right)\right)}.$$
- So growth rates are stratified by depth.

The exponential range

- The next natural range is if $f_M(n) = o(c^n)$ for some $c \in \mathbb{R}$.
- Here we expect the structures to be "tree-like".
- We say M is *NIP* if it does not encode arbitrary bipartite graphs (this is very closely related to bounded VC-dimension), and *monadically NIP* if it remains NIP after an arbitrary coloring of its elements.

Conjecture

Let M be a countable homogeneous structure, and $\mathcal{C}_M$ its class of finite substructures. Then the following are equivalent.

1. *M is monadically NIP.*
2. $f_M(n) = o(c^n)$ *for some* $c \in \mathbb{R}$.
3. $\mathcal{C}_M$ *has no infinite antichains under embeddability.*
4. $\mathcal{C}_M$ *is algorithmically tractable.*

- See recent work on twin-width and sparse graph classes.

Beyond homogeneity

Question

What can we prove if the homogeneity assumption is removed? What about arbitrary hereditary classes?

Conjecture

Let C be a hereditary class of (unlabelled) structures in a finite relational language, and let C_n be the subclass of structures of size n. Either $|C_n| \sim cn^k$ for some $c \in \mathbb{R}$ and $k \in \mathbb{N}$, or $|C_n|$ grows at least as fast as the partition function.

- This conjecture is known for hereditary graph classes [1].

References I

[1] József Balogh, Béla Bollobás, Michael Saks, and Vera T Sós, *The unlabelled speed of a hereditary graph property*, Journal of Combinatorial Theory, Series B **99** (2009), no. 1, 9–19.

[2] Samuel Braunfeld, *Monadic stability growth rates of ω-categorical structures*, arXiv preprint arXiv:1910.04380 (2019).

[3] Peter J Cameron, *Oligomorphic permutation groups*, Perspectives in mathematical sciences II: Pure mathematics, 2009, pp. 37–61.